Prove that the sequence $(a_n)_{n\geq 1}$ given by $a_n = (-1)^n$ does not exist.

Scratch work.

$(\lim a_n \text{ exists}) \iff \exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, n \in \mathbb{N}$, $|a_n - L| < \varepsilon$.

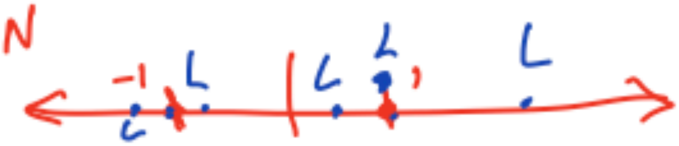
$(\lim a_n \text{ does not exist}) \iff$ $\forall L \in \mathbb{R}$, $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n \in \mathbb{N}$ s.t. $n \geq N$ with $|a_n - L| \geq \varepsilon$.

I will find ε (depends on L)

I will find n, depends on L, ε, N.

goal $|a_n - L| \geq \varepsilon$

(pick n s.t. $n \geq N$) $|(-1)^n - L| \geq 1$ choose if $L > 0$ choose n odd if L is ≤ 0 choose n even



Actual proof. $\forall L \in \mathbb{R}$, choose $\varepsilon = 1 > 0$.
$\forall N \in \mathbb{N}$, let n be $\geq N$ and odd if $L > 0$ and let n be $\geq N$ and even if $L \leq 0$. Then

$$\left|(-1)^n - L\right| \geq 1,$$

so $\lim_{n\to\infty} (-1)^n$ does not exist. $\boxtimes$

Example Prove that

$$\lim \frac{3k^2 - (-1)^k}{k^2} \text{ exists.}$$

Scratch. Limit $= L = 3$.

Want $\left| \frac{3k^2 - (-1)^k}{k^2} - 3 \right| < \varepsilon$

$$\Updownarrow \quad \left| \frac{3k^2 - (-1)^k - 3k^2}{k^2} \right| < \varepsilon$$

$$\Updownarrow \quad \left| -\frac{(-1)^k}{k^2} \right| < \varepsilon$$

$$\frac{|(-1)^k|}{k^2} < \varepsilon$$

$$\frac{1}{k^2} < \varepsilon$$

$$1 < \varepsilon k^2$$

$$\frac{1}{\varepsilon} < k^2$$

$$\sqrt{\frac{1}{\varepsilon}} < k \qquad \text{both +}$$

$$1 + \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil = N.$$

actual proof. $\forall \varepsilon > 0$

Let $N = 1 + \lceil \sqrt{\frac{1}{\varepsilon}} \rceil$. Then

$\forall k \geq N$,

$$\left| \frac{3k^2 - (-1)^k}{k^2} - 3 \right| = \left| \frac{3k^2 - (-1)^k - 3k^2}{k^2} \right|$$

$$= \frac{1}{k^2} \leq \frac{1}{\left(1 + \lceil \sqrt{\frac{1}{\varepsilon}} \rceil\right)^2}$$

$$< \frac{1}{\left(\lceil \sqrt{\frac{1}{\varepsilon}} \rceil\right)^2} \leq \frac{1}{\left(\sqrt{\frac{1}{\varepsilon}}\right)^2}$$

$$= \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

Thus $\lim \frac{3k^2 - (-1)^k}{k^2} = 3$. $\blacksquare$

Example

Let $b_1 = \sqrt{4} = 2$ ☆

$b_2 = \sqrt{4+\sqrt{4}}$

$b_3 = \sqrt{4+\sqrt{4+\sqrt{4}}}$

In other words, let

$$b_{n+1} = \sqrt{4+b_n}$$ ☆

Does this converge?
If so, what is the limit?

We need some tools!!!!!

1. Algebraic Limit Theorem.

Suppose

$\lim a_n$ exists

$\lim b_n$ exists

Then

a. $\lim (a_n + b_n) = \lim (a_n) + \lim (b_n)$

b. $\lim (a_n - b_n) = \lim (a_n) - \lim (b_n)$

c. $\lim (a_n \cdot b_n) = \lim (a_n) \lim (b_n)$

d. $\lim (a_n / b_n) = \dfrac{\lim (a_n)}{\lim (b_n)}$

(if $b_n \neq 0\ \forall n$, $\lim b_n \neq 0$.)

(e) $\lim \sqrt{a_n} = \sqrt{\lim(a_n)}$

(if $a_n \geq 0\ \forall n$ and $\lim(a_n) \geq 0$).

Proof of (b)

Scratch: $\lim a_n = A$

$\lim b_n = B$

wait $\left| a_n - b_n - \underbrace{(A - B)}_{-A+B} \right| < \varepsilon$

$\Updownarrow$

$\left| a_n - A - b_n + B \right| < \varepsilon$

$\Updownarrow$

$\boxed{\left| (a_n - A) - (b_n - B) \right|} < \varepsilon$

$\Uparrow$

$$|a_n - An| + |b_n - B| < \varepsilon$$

$$\frac{\varepsilon}{2} \qquad \frac{\varepsilon}{2}.$$

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actual proof of (b). $\forall \varepsilon > 0$,

Since $\lim a_n = A$ and $\lim b_n = B$,

$\exists N_1 \in \mathbb{N}$ s.t. $\forall n_1 \geq N_1$,

$|a_{n_1} - A| < \frac{\varepsilon}{2}$ and

$\exists N_2 \in \mathbb{N}$ s.t. $\forall n_2 \geq N_2$

$|b_{n_2} - B| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$.

Then $\forall n \geq N$, $|a_n - A| < \frac{\varepsilon}{2}$ and

$|b_n - B| < \varepsilon/2$.

Then $\left|(a_n - b_n) - (A - B)\right|$

$$= \left|(a_n - A) - (b_n - B)\right|$$

$$\leq |a_n - A| + |b_n - B|$$

by the triangle ineq.

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus

$$\lim (a_n - b_n) = A - B. \blacksquare$$